Asymptotic behaviour for Willmore surfaces of revolution under natural boundary conditions

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1 Introduction

For a smooth, immersed surface $\Gamma \subset \mathbb{R}^3$ and a parameter $\gamma \in [0,1]$ we consider the functional

$$W_\gamma(\Gamma) := \int_\Gamma H^2 dA - \gamma \int_\Gamma K dA,$$

where H is the mean curvature of the immersion and K its Gauss curvature. This functional models the elastic energy of thin shells. Willmore studied in [7] the functional W_0, by now called *Willmore functional*.

First we note that $W_\gamma(\Gamma) \geq 0$ holds for every $\gamma \in [0,1]$. Let $\kappa_1, \kappa_2 \in \mathbb{R}$ denote the principal curvatures of the surface. Then $H^2 - \gamma K = \frac{1}{4}(\kappa_1 + \kappa_2)^2 - \gamma \kappa_1 \cdot \kappa_2 = \frac{1-\gamma}{4}(\kappa_1 + \kappa_2)^2 + \frac{\gamma}{4}(\kappa_1 - \kappa_2)^2 \geq 0$ for $\gamma \in [0,1]$ gives the semi-definiteness. We are interested in minima or critical points of W_γ. Such critical points have to satisfy the *Willmore equation*

$$\Delta_\Gamma H + 2H\left(H^2 - K\right) = 0 \quad \text{on} \quad \Gamma, \tag{1}$$

where Δ_Γ denotes the Laplace-Beltrami operator on Γ. A solution of this equation is called *Willmore surface*. For references and further background information we refer to Nitsche's survey article [6].

In order to present a complete analysis of special Willmore surfaces satisfying prescribed boundary conditions, we restrict ourselves to surfaces of revolution generated by rotating a symmetric graph $u : [-1,1] \to (0,\infty)$ about the x-axis. Existence and classical regularity of those axially symmetric Willmore surfaces with arbitrary symmetric Dirichlet boundary conditions were proved in [3] and [4]. Furthermore in [1] Bergner, Dall'Acqua and Fröhlich solved the existence problem for Willmore surfaces of revolution with prescribed position at the boundary while the second boundary condition is a natural one. It arises when considering critical points of the Willmore surface in the class of surfaces of revolution generated by symmetric graphs where only the position at the boundary is fixed. All these existence results have been obtained by minimising the Willmore functional in suitable classes of admissible functions. We will show that the energy minimising solutions under natural boundary conditions converge to the sphere for $u(\pm 1) \searrow 0$. Such a result was previously proved in [4] for solutions of the Dirichlet problem.

1.1 Main result

We define some real number α^* by

$$\alpha^* := \min_{y>0} \frac{\cosh(y)}{y} = \frac{1}{b^*}\cosh(b^*) \approx 1.5088795... \tag{2}$$

$$\text{with } \; b^* \approx 1.1996786... \; \text{ solving } \; b^* \tanh(b^*) = 1. \tag{3}$$

Then the main result is the following.

Theorem 1.1. *For each $\gamma \in [0,1]$ and for $\alpha > 0$ such that $\alpha^* > \alpha$ let $u_\alpha \in C^\infty([-1,1],(0,\infty))$ be a positive and symmetric minimiser of the Willmore functional*

$$W_\gamma(u_\alpha) = \inf \{W_\gamma(v) : v \in N_\alpha\}$$

in the class

$$N_\alpha := \left\{ v \in H^2((-1,1),(0,\infty)) : v(x) = v(-x), v(\pm 1) = \alpha \right\}.$$

In particular, the corresponding surface of revolution $\Gamma \subset \mathbb{R}^3$ solves

$$\begin{cases} \Delta_\Gamma H + 2H(H^2 - K) = 0 \quad \text{on} \;\; \Gamma, \\ \\ u_\alpha(\pm 1) = \alpha \quad \text{and} \;\; H(\pm 1) = \frac{\gamma}{\alpha\sqrt{1+u_\alpha'(\pm 1)^2}}. \end{cases} \tag{4}$$

Furthermore let u_0 denote the semicircle $u_0(x) = \sqrt{1-x^2}$, $x \in [-1,1]$. Then, for any $m \in \mathbb{N}$,

$$\lim_{\alpha \searrow 0} u_\alpha = u_0 \quad \text{in} \;\; C^m_{loc}(-1,1).$$

2 Existence results

2.1 Surfaces of revolution

We consider functions $u \in C^4([-1,1],(0,\infty))$. A surface of revolution $\Gamma \subset \mathbb{R}^3$ can be parametrised by

$$\Gamma: \ (x,\phi) \mapsto (x, u(x)\cos(\phi), u(x)\sin(\phi)), \ x \in [-1,1], \phi \in [0,2\pi].$$

Let κ_1 and κ_2 denote the principal curvatures of $\Gamma \subset \mathbb{R}^3$. Its mean curvature H and Gauss curvature K are given by

$$H = \frac{\kappa_1+\kappa_2}{2} = -\frac{u''(x)}{2\left(1+u'(x)^2\right)^{\frac{3}{2}}} + \frac{1}{2u(x)\left(1+u'(x)^2\right)^{\frac{1}{2}}},$$
$$K = \kappa_1\kappa_2 = -\frac{u''(x)}{u(x)\left(1+u'(x)^2\right)^2},$$

respectively. Then $W_\gamma(\Gamma)$ takes the form

$$W_\gamma(u) := W_\gamma(\Gamma) = \frac{\pi}{2}\int_{-1}^{1}\left(\frac{u''(x)}{\left(1+u'(x)^2\right)^{\frac{3}{2}}} - \frac{1}{u(x)\sqrt{1+u'(x)^2}}\right)^2 u(x)\sqrt{1+u'(x)^2}dx$$
$$+ 2\pi\gamma\left[\frac{u'(x)}{\sqrt{1+u'(x)^2}}\right]_{-1}^{1}.$$

2.2 Hyperbolic Willmore functional

For our purposes it will be convenient to consider the profiles of surfaces of revolution also as curves in the hyperbolic plane. We benefit from observations made by Bryant, Griffiths and Pinkall (see e.g. [2, 5]). We consider the hyperbolic plane $\mathbb{R}^2_+ := \{(x,y) : y > 0\}$ with the metric

$$ds^2 = \frac{1}{y^2}\left(dx^2 + dy^2\right).$$

The associated hyperbolic curvature is given by

$$\kappa_h(x) = \frac{1}{\left(1+u'(x)^2\right)^{\frac{1}{2}}} + \frac{u(x)u''(x)}{\left(1+u'(x)^2\right)^{\frac{3}{2}}}.$$

For $u \in H^2((-1,1),(0,\infty))$ the *hyperbolic Willmore functional* is defined by

$$W_h(u) := \int_{-1}^{1}\kappa_h(x)^2 ds(x) = \int_{-1}^{1}\left(\frac{1}{\left(1+u'(x)^2\right)^{\frac{1}{2}}} + \frac{u(x)u''(x)}{\left(1+u'(x)^2\right)^{\frac{3}{2}}}\right)^2 \frac{\sqrt{1+u'(x)^2}}{u(x)}dx.$$

The advantage of the hyperbolic plane is that the hyperbolic curvature of an arc of a circle with centre on the x-axis is equal to zero. Furthermore $W_h(u)$ is equal to zero if and only if $u(x) = \sqrt{r^2 - x^2}$ for $x \in [-1,1]$ and $r \geq 1$.

Comparing our two functionals we see that for each $u \in H^2((-1,1),(0,\infty))$ and each $\gamma \in [0,1]$ it holds that

$$W_\gamma(u) = \frac{\pi}{2}W_h(u) + 2\pi(\gamma-1)\left[\frac{u'(x)}{\sqrt{1+u'(x)^2}}\right]_{-1}^{1}. \tag{5}$$

This shows that both functionals are equivalent as long as $u'(-1)$ and $u'(1)$ are kept fixed.

Remark 2.1. An important property of the energy W_h is its rescaling invariance, i.e. given a positive $u \in H^2((-r,r),(0,\infty))$ for some $r > 0$, then the rescaled function $v := \frac{1}{r}u(r\,.) \in H^2((-1,1),(0,\infty))$ has the same energy as u, that is,

$$W_h(v) = \int_{-r}^{r} \kappa_h^2[u]ds[u].$$

Since

$$\left[\frac{v'(x)}{\sqrt{1+v'(x)}}\right]_{-1}^{1} = \left[\frac{u'(rx)}{\sqrt{1+u'(rx)}}\right]_{-1}^{1} = \left[\frac{u'(x)}{\sqrt{1+u'(x)}}\right]_{-r}^{r},$$

also the energy W_γ is invariant under rescaling. This is a particular case of the well-known conformal invariance of the Willmore functional W_γ.

2.3 Existence results

Definition 2.2. *For $\alpha > 0$, $\beta \in \mathbb{R}$ and $\gamma \in [0,1]$ we define*

$$\begin{aligned} &N_{\alpha,\beta} := \left\{u \in H^2((-1,1),(0,\infty)) : \ u(x) = u(-x), \ u(\pm 1) = \alpha, \ u'(-1) = \beta\right\}, \\ &M^h_{\alpha,\beta} := \inf\left\{W_h(u) : \ u \in N_{\alpha,\beta}\right\}, \ M_{\gamma,\alpha,\beta} := \inf\left\{W_\gamma(u) : \ u \in N_{\alpha,\beta}\right\} \quad \textit{and} \\ &M_{\gamma,\alpha} := \inf_{\beta\in\mathbb{R}} M_{\gamma,\alpha,\beta} = \inf\{W_\gamma(u) : \ u \in N_\alpha\}. \end{aligned}$$

One easily sees that $N_{\alpha,\beta}$ is never empty, hence $M^h_{\alpha,\beta}$, $M_{\gamma,\alpha,\beta}$, and $M_{\gamma,\alpha}$ are well defined. We recall the existence result for the Dirichlet boundary value problem (6) below from [4]. This result holds true for $\gamma = 0$ and $\gamma = 1$. For β fixed (5) shows that a solution to this problem is a critical point for W_γ independently of γ.

Theorem 2.3. *(See [4, Theorem 1.1].) For each $\alpha > 0$ and $\beta \in \mathbb{R}$, there exists a positive and symmetric function $u \in C^\infty([-1,1],(0,\infty))$ such that the corresponding surface of revolution $\Gamma \subset \mathbb{R}^3$ solves*

$$\begin{cases} \Delta_\Gamma H + 2H(H^2 - K) = 0 \quad \textit{on} \ \ \Gamma, \\ \\ u(\pm 1) = \alpha \quad \textit{and} \ \ u'(-1) = -u'(1) = \beta, \end{cases} \tag{6}$$

and satisfies $W_h(u) = M^h_{\alpha,\beta}$.

In order to prove Theorem 1.1 we also have to recall the properties of energy minimising solutions for α small. First, [4, Lemma 5.1, Part 1] shows the following.

Lemma 2.4. *For each $\alpha > 0$ and $\beta \geq 0$ such that $\alpha\beta \leq 1$ let u be a solution to problem (6) such that $W_h(u) = M^h_{\alpha,\beta}$. Then, $u \in C^\infty([-1,1],(0,\infty))$ and u has the following properties.*

$$x + u(x)u'(x) > 0 \quad \textit{in} \ \ (0,1) \quad \textit{and} \ \ u'(x) < 0 \ \textit{in} \ (0,1).$$

Furthermore, [4, Lemma 5.1, Part 2] provides the next Lemma.

Lemma 2.5. *For $\alpha < \alpha^*$ and $\beta < 0$ let u be a solution to problem (6) such that $W_h(u) = M^h_{\alpha,\beta}$. Then, $u \in C^\infty([-1,1],(0,\infty))$ and u has the following properties.*

$$x + u(x)u'(x) > 0 \textit{ in } (0,1) \quad \textit{and} \quad u'(x) \le \max\{\alpha^*, -\beta\} \textit{ in } [0,1].$$

There exists at most one point $x_0 \in (0,1)$ with $u'(x_0) = 0$.

Taking advantage of the previous results the existence problem for (4) was solved in [1].

Theorem 2.6. *(See [1, Theorem 1.1].) For each $\alpha > 0$ and each $\gamma \in [0,1]$, there exists a positive and symmetric function $u \in C^\infty([-1,1],(0,\infty))$, i.e. $u(x) > 0$ and $u(x) = u(-x)$, such that the corresponding surface of revolution $\Gamma \subset \mathbb{R}^3$ solves*

$$\begin{cases} \Delta_\Gamma H + 2H(H^2 - K) = 0 \quad \textit{on} \quad \Gamma, \\ \\ u(\pm 1) = \alpha \quad \textit{and} \quad H(\pm 1) = \frac{\gamma}{\alpha\sqrt{1+u'(\pm 1)^2}} \end{cases}$$

and satisfies $W_\gamma(u) = M_{\gamma,\alpha}$.

We restrict ourselves to $\alpha < \alpha^*$ and show some properties of energy minimising solutions to (4).

Lemma 2.7. *For $\alpha < \alpha^*$ and $\gamma \in [0,1]$ let u be a solution to problem (4) such that $W_\gamma(u) = M_{\gamma,\alpha}$. Then, $u \in C^\infty([-1,1],(0,\infty))$ and u has the following properties:*

$$-\alpha \le u'(-1) \le \alpha^{-1}, \; x + u(x)u'(x) \ge 0 \quad \textit{in} \quad [0,1] \quad \textit{and} \quad u'(x) \le \alpha^* \quad \textit{in} \quad [0,1]. \qquad (7)$$

There exists at most one point $x_0 \in (0,1)$ with $u'(x_0) = 0$.

Proof. The monotonicity of the mapping $\beta \mapsto M_{\gamma,\alpha,\beta}$ (see [1, Corollary 3.13]) shows $W_\gamma(u) = M_{\gamma,\alpha} = \inf_{-\alpha\le\beta\le\alpha^{-1}} M_{\gamma,\alpha,\beta}$. Therefore we have $u'(-1) \in [-\alpha, \alpha^{-1}]$ and u minimises W_γ in the class $N_{\alpha,u'(-1)}$, i.e. $W_\gamma(u) = M_{\gamma,\alpha,u'(-1)}$. Because of (5) u is also a critical point for W_h, hence $W_h(u) = M^h_{\alpha,u'(-1)}$. Now if $u'(-1) \in [0, \alpha^{-1}]$ Lemma 2.4 provides (7). If $u'(-1) \in [-\alpha, 0)$ notice that $-u'(-1) \le \alpha < \alpha^*$. Hence Lemma 2.5 shows (7). □

3 Convergence to the sphere for $\alpha \searrow 0$

The proof of Theorem 1.1 is quite similar to [4, Chapter 5]. The main difference is that we have no fixed β. When we consider a sequence α_k and corresponding solutions u_{α_k} to problem (4) we have no control of the sign of $u'_{\alpha_k}(-1)$. Therefore we have to consider subsequences α_{k_ℓ} such that $u'_{\alpha_{k_\ell}}(-1)$ is either positive or negative for all ℓ.

Lemma 3.1. *Let $(\alpha_k)_{k\in\mathbb{N}} \subset \mathbb{R}$ be a sequence such that $\alpha_k > 0$ for all $k \in \mathbb{N}$ and $\lim_{k\to\infty} \alpha_k = 0$. For $\gamma \in [0,1]$ we assume $u_{\alpha_k} \in N_{\alpha_k}$ minimises the energy W_γ, i.e. $W_\gamma(u_{\alpha_k}) = M_{\gamma,\alpha_k}$ for all $k \in \mathbb{N}$. If there is a subsequence $(\alpha_{k_\ell})_{\ell\in\mathbb{N}}$ such that $\beta(\alpha_{k_\ell}) := u'_{\alpha_{k_\ell}}(-1) < 0$, then there exists a sequence $(x_{\alpha_{k_\ell}})_{\ell\in\mathbb{N}} \subset [0,1)$ with the following properties:*

$$u'_{\alpha_{k_\ell}}(x_{\alpha_{k_\ell}}) = 0,\ u'_{\alpha_{k_\ell}} > 0 \quad \textit{in} \quad (x_{\alpha_{k_\ell}}, 1] \quad \textit{and} \quad \lim_{\ell\to\infty} x_{\alpha_{k_\ell}} = 1.$$

Proof. Let $(\alpha_k)_{k\in\mathbb{N}} \subset \mathbb{R}$ be given and assume there is a subsequence $(\alpha_{k_\ell})_{\ell\in\mathbb{N}}$ such that $\beta(\alpha_{k_\ell}) := u'_{\alpha_{k_\ell}}(-1) < 0$. One easily sees that a sequence $(x_{\alpha_{k_\ell}})_{\ell\in\mathbb{N}} \subset [0,1)$ exists satisfying $u'_{\alpha_{k_\ell}}(x_{\alpha_{k_\ell}}) = 0$ and $u'_{\alpha_{k_\ell}} > 0$ in $(x_{\alpha_{k_\ell}}, 1]$ because of $u'_{\alpha_{k_\ell}}(1) > 0$ and the symmetry of the $u_{\alpha_{k_\ell}}$. It remains to show that $\lim_{\ell\to\infty} x_{\alpha_{k_\ell}} = 1$ holds. One can do this by using the arguments of the proof to [4, Lemma 5.2]. Just notice that every $u_{\alpha_{k_\ell}}$ also minimises the energy W_h in the class $N_{\alpha,\beta(\alpha_{k_\ell})}$, i.e. $W_h(u_{\alpha_{k_\ell}}) = M^h_{\alpha,\beta(\alpha_{k_\ell})}$ and for all $\ell \in \mathbb{N}$ it holds that

$$\frac{\beta(\alpha_{k_\ell})}{\sqrt{1+\beta(\alpha_{k_\ell})^2}} \in [-1,0]. \qquad \square$$

The next Lemma is an analogue to [4, Lemma 5.3]. The proofs are similar except the fact that after choosing the sequence $(\alpha_k)_{k\in\mathbb{N}}$ we have to pass to a subsequence as above.

Lemma 3.2. *We fix $\delta_0 \in (0,1)$. For $\alpha > 0$ and $\gamma \in [0,1]$, let $u_\alpha \in N_\alpha$ be a minimiser of the energy W_γ, i.e. $W_\gamma(u_\alpha) = M_{\gamma,\alpha}$. Then,*

$$\lim_{\alpha\searrow 0} \max_{x\in[-1,-1+\delta_0]} u'_\alpha(x) = \infty.$$

Theorem 3.3. *For $\alpha > 0$ and $\gamma \in [0,1]$ let $u_\alpha \in N_\alpha$ be such that $W_\gamma(u_\alpha) = M_{\gamma,\alpha}$. Then, for all $\delta_0 \in (0,1)$*

$$\lim_{\alpha\searrow 0} \int_{-1+\delta_0}^{1-\delta_0} \kappa_h[u_\alpha]^2 \frac{\sqrt{1+u'_\alpha(x)^2}}{u_\alpha(x)} dx = 0.$$

Proof. For any sequence $\alpha_k \searrow 0$, by Lemma 3.2 there exists $\delta_{\alpha_k} \in [0,\delta_0]$ with $\lim_{k\to\infty} u'_{\alpha_k}(1-\delta_{\alpha_k}) = +\infty$. We define $\beta_k := u'_{\alpha_k}(-1)$ for all $k \in \mathbb{N}$ and, observing that $-\alpha_k \le \beta_k \le \frac{1}{\alpha_k}$,

$$f_{\alpha_k}(x) := \begin{cases} \frac{\alpha_k}{\sqrt{1+\beta_k^2}} \cosh\left(\frac{\sqrt{1+\beta_k^2}}{\alpha_k}(x-x_1)\right) & : \ x_0 \le x \le 1 \\ \sqrt{r^2-x^2} & : \ -x_0 < x < x_0 \\ \frac{\alpha_k}{\sqrt{1+\beta_k^2}} \cosh\left(\frac{\sqrt{1+\beta_k^2}}{\alpha_k}(x+x_1)\right) & : \ -1 \le x \le -x_0, \end{cases}$$

where $x_1 = 1 - \alpha_k \operatorname{arsinh}(-\beta_k)/\sqrt{1+\beta_k^2}$, $r^2 = x_0^2 + u_{\alpha_k}(x_0)^2$ and, for α_k small enough, $x_0 \in (0,1)$ is the solution of

$$-x_0 = \frac{\alpha_k}{2\sqrt{1+\beta_k^2}} \sinh\left(\frac{2\sqrt{1+\beta_k^2}}{\alpha_k}(x_0-x_1)\right).$$

By [4, Section 5.1] the function f_{α_k} is in N_{α_k,β_k} and has hyperbolic Willmore energy

$$-8\frac{\beta_k}{\sqrt{1+\beta_k^2}}+8 \geq W_h(f_{\alpha_k}).$$

u_{α_k} minimises the energy W_h in the class N_{α_k,β_k}, hence for all $k \in \mathbb{N}$

$$\begin{aligned}-8\frac{\beta_k}{\sqrt{1+\beta_k^2}}+8 \geq W_h(f_{\alpha_k}) \geq W_h(u_{\alpha_k}) \geq &\int_{-1+\delta_{\alpha_k}}^{1-\delta_{\alpha_k}} \kappa_h[u_{\alpha_k}]^2 \frac{\sqrt{1+u_{\alpha_k}'(x)^2}}{u_{\alpha_k}(x)}dx \\ &+8\frac{u_{\alpha_k}'(-1+\delta_{\alpha_k})}{\sqrt{1+u_{\alpha_k}'(-1+\delta_{\alpha_k})^2}} - 8\frac{\beta_k}{\sqrt{1+\beta_k^2}}.\end{aligned}$$

It follows for all $k \in \mathbb{N}$

$$\begin{aligned}8-8\frac{u_{\alpha_k}'(-1+\delta_{\alpha_k})}{\sqrt{1+u_{\alpha_k}'(-1+\delta_{\alpha_k})^2}} &\geq \int_{-1+\delta_{\alpha_k}}^{1-\delta_{\alpha_k}} \kappa_h[u_{\alpha_k}]^2 \frac{\sqrt{1+u_{\alpha_k}'(x)^2}}{u_{\alpha_k}(x)}dx \\ &\geq \int_{-1+\delta_0}^{1-\delta_0} \kappa_h[u_{\alpha_k}]^2 \frac{\sqrt{1+u_{\alpha_k}'(x)^2}}{u_{\alpha_k}(x)}dx \geq 0.\end{aligned}$$

Then, $\lim_{k\to\infty} u_{\alpha_k}'(1-\delta_{\alpha_k}) = +\infty$ shows the claim.

□

The next lemma corresponds to [4, Lemma 5.6]. Again the proofs are similar except the fact that after choosing the sequence $(\alpha_k)_{k\in\mathbb{N}}$ we have to pass to a subsequence such that the derivatives at -1 of the corresponding solutions to (4) are either all positive or all negative.

Lemma 3.4. *Fix $\delta_0 \in (0,1)$. For $\alpha > 0$ and $\gamma \in [0,1]$ let $u_\alpha \in N_\alpha$ solve $W_\gamma(u_\alpha) = M_{\gamma,\alpha}$. Then, there exists $\varepsilon > 0$ such that $u_\alpha(x) \geq \varepsilon$ in $[-1+\delta_0, 1-\delta_0]$ for all $\alpha \leq 1$.*

Lemma 3.5. *Fix $\delta_0 \in (0,1)$. For $\gamma \in [0,1]$ and $\alpha > 0$ small enough let $u_\alpha \in N_\alpha$ solve $W_\gamma(u_\alpha) = M_{\gamma,\alpha}$. Then, there exists $\varepsilon > 0$ such that*

$$-\frac{1}{\varepsilon} \leq u_\alpha'(x) \leq \alpha^* \quad \text{for all} \quad x \in [0, 1-\delta_0].$$

Proof. By Lemma 2.7 we have $x+u_\alpha(x)u_\alpha'(x) > 0$ in $[0,1]$. Together with Lemma 3.4 this shows the first inequality. The second one follows from the estimates on the minimiser in Lemma 2.7.

□

Now, using Lemmas 3.4 and 3.5 the proof of Theorem 1.1 is similar to the one of [4, Theorem 5.8].

References

[1] M. Bergner, A. Dall'Acqua, and S. Fröhlich. Symmetric Willmore surfaces of revolution satisfying natural boundary conditions. *Calc. Var. Partial Differential Equations*, to appear, 2010. http://dx.doi.org/10.1007/s00526-010-0313-7.

[2] R. Bryant and P. Griffiths. Reduction for constrained variational problems and $\int \frac{1}{2}k^2\,ds$. *Amer. J. Math.*, 108:525–570, 1986.

[3] A. Dall'Acqua, K. Deckelnick, and H.-Ch. Grunau. Classical solutions to the Dirichlet problem for Willmore surfaces of revolution. *Advances in Calculus of Variations*, 1:379–397, 2008.

[4] A. Dall'Acqua, S. Fröhlich, H.-Ch. Grunau, and F. Schieweck. Symmetric Willmore surfaces of revolution satisfying arbitrary Dirichlet boundary data. *Preprint 24-2008, Universität Magdeburg, Fakultät für Mathematik*, 2008.

[5] U. Hertrich-Jeromin and U. Pinkall. Ein Beweis der Willmoreschen Vermutung für Kanaltori. *J. Reine Angew. Math.*, 430:21–34, 1992.

[6] J.C.C. Nitsche. Boundary value problems for variational integrals involving surface curvatures. *Quart. Appl. Math*, 51(2), 1993.

[7] T.J. Willmore. *Riemannian geometry.* Oxford University Press, USA, 1996.