

A decomposition method with respect to dual cones and its application to higher order Sobolev spaces

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Abstract

In this seminar paper we study a decomposition method with respect to dual cones, which was established by J. J. Moreau. Moreover, a couple of explicit examples are presented, helping to observe a connection with obstacle problems. In the last section we will give some well known results on this problem.

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1 Abstract decomposition method

First we will repeat some basic results of linear functional analysis (see [1]), which help to simplify the understanding of the decomposition of Moreau. In the following we denote by \mathcal{H} a real Hilbert space endowed with the scalar product $\langle \cdot, \cdot \rangle_{\mathcal{H}}$.

Theorem 1 (projection theorem)

Let \mathcal{H} be a Hilbert space and \mathcal{K} a non-empty closed convex subset of \mathcal{H} . Then there exists for any $x \in \mathcal{H}$ a unique $y \in \mathcal{K}$ with

$$\|x - y\| = \inf_{z \in \mathcal{K}} \|x - z\|. \quad (1)$$

Furthermore, for y the following characterization holds:

$$\langle x - y, z - y \rangle_{\mathcal{H}} \leq 0 \quad \forall z \in \mathcal{K}. \quad (2)$$

PROOF: At first we show the existence of such a y .

We consider an arbitrary $x \in \mathcal{H}$ and define $d := \inf_{z \in \mathcal{K}} \|x - z\|$.

Then there exists a sequence $(z_k)_{k \in \mathbb{N}} \subset \mathcal{K}$ with:

$$\|x - z_k\| \searrow d, \quad k \rightarrow \infty \quad (3)$$

Since \mathcal{H} is a Hilbert space and consequently the parallelogram identity

$$\|(x - z_k) - (x - z_l)\|^2 + \|(x - z_k) + (x - z_l)\|^2 = 2(\|x - z_k\|^2 + \|x - z_l\|^2)$$

is valid, we have

$$\|z_k - z_l\|^2 = 2(\|x - z_k\|^2 + \|x - z_l\|^2) - 4\|x - \frac{z_k + z_l}{2}\|^2.$$

Because of the convexity of \mathcal{K} we also have $\frac{z_k + z_l}{2} \in \mathcal{K}$ and therefore $\|x - \frac{z_k + z_l}{2}\| \geq d$ holds. This and (3) yield

$$\|z_k - z_l\|^2 \leq 2(d^2 + d^2) + o(1) - 4d^2 \xrightarrow{k, l \rightarrow \infty} 0,$$

that means $(z_k)_{k \in \mathbb{N}}$ is a Cauchy sequence. Taking into account that \mathcal{H} is complete and \mathcal{K} is closed, we can conclude that $z_k \xrightarrow{k \rightarrow \infty} y$ with some suitable $y \in \mathcal{K}$. That y satisfies equation (1), follows from

$$d = \lim_{k \rightarrow \infty} \|x - z_k\| = \|x - \lim_{k \rightarrow \infty} z_k\| = \|x - y\|$$

proving the existence part.

We still have to prove uniqueness of y . Let y_1 also satisfy equation (1). Then we have

$$\begin{aligned}
0 \leq \|y - y_1\|^2 &= \|(x - y_1) - (x - y)\|^2 \\
&= 2(\underbrace{\|x - y\|^2}_{=d} + \underbrace{\|x - y_1\|^2}_{=d}) - 4\| \underbrace{x - \frac{y - y_1}{2}}_{\in \mathcal{K}} \|^2 \\
&\leq 4d^2 - 4d^2 = 0.
\end{aligned}$$

that is $\|y - y_1\| = 0$, giving $y = y_1$.

In order to complete the proof, we have to verify the characterization (2) of y . Let y satisfy equation (1).

We chose $z \in \mathcal{K}$ and $\alpha \in (0, 1)$ so that $(1 - \alpha)y + \alpha z \in \mathcal{K}$ holds. Then we obtain

$$\begin{aligned}
\|x - y\|^2 &\leq \|x - (1 - \alpha)y - \alpha z\|^2 = \|x - y - \alpha(z - y)\|^2 \\
&= \|x - y\|^2 - 2\alpha \langle x - y, z - y \rangle_{\mathcal{H}} + \alpha^2 \|z - y\|^2
\end{aligned}$$

and consequently

$$2\langle x - y, z - y \rangle_{\mathcal{H}} \leq \alpha^2 \|z - y\|^2.$$

With $\alpha \searrow 0$ it follows that inequality (2) fulfilled.

Now, we assume that inequality (2) holds. We obtain

$$\begin{aligned}
\|x - z\|^2 &= \|x - y + y - z\|^2 \\
&= \|x - y\|^2 + 2 \underbrace{\langle x - y, y - z \rangle_{\mathcal{H}}}_{\geq 0} + \|y - z\|^2 \\
&\geq \|x - y\|^2, \quad \forall z \in \mathcal{K}
\end{aligned}$$

showing that y minimizes the distance. So it satisfies equation (1). \square

Remark: The above y specified by equation (1), is called the projection of x upon \mathcal{K} and will be denoted by $y = \text{proj}_{\mathcal{K}} x$.

After presenting this important result, we will now continue with the geometrical objects used in this paper.

Definition 2 (cone and dual cone)

(i) A set $\mathcal{K} \subseteq \mathcal{H}$ is called a cone, if for all $x \in \mathcal{K}$ and all $\alpha \in \mathbb{R}^+$ also $\alpha \cdot x \in \mathcal{K}$ holds.

(ii) Let $\mathcal{K} \subseteq \mathcal{H}$ be a cone. Then the set $\mathcal{K}^* := \{x \in \mathcal{H} \mid \forall y \in \mathcal{K} : \langle x, y \rangle_{\mathcal{H}} \leq 0\}$ is called the dual cone of \mathcal{K} .

(iii) Let $\mathcal{C}, \mathcal{K} \subseteq \mathcal{H}$ be two cones. Then we say \mathcal{C} and \mathcal{K} are mutually dual, if both $\mathcal{K} = \mathcal{C}^*$ and $\mathcal{C} = \mathcal{K}^*$ holds.

Theorem 3 Let $\mathcal{K} \subset \mathcal{H}$ be a closed convex cone. Any $x \in \mathcal{H}$ can be decomposed in the form

$$x = y + z \quad \text{with} \quad \langle z, y \rangle_{\mathcal{H}} = 0,$$

where $y \in \mathcal{K}$ and $z \in \mathcal{K}^*$.

PROOF: Let $x \in \mathcal{H}$. Moreover, let $y = \text{proj}_{\mathcal{K}}x$ and $z = x - y$. Because of equation (2) of Theorem 1

$$\langle x - y, p - y \rangle_{\mathcal{H}} \leq 0 \quad (4)$$

holds for all p in \mathcal{K} . If we now choose $p = \lambda y$, $\lambda \geq 0$, it follows that

$$(\lambda - 1)\langle z, y \rangle_{\mathcal{K}} \leq 0.$$

Since $\lambda \geq 0$ is arbitrary, we have $\langle z, y \rangle_{\mathcal{H}} = 0$. This combined with (4) yields

$$\langle z, p \rangle_{\mathcal{H}} \leq 0, \quad \forall p \in \mathcal{K}.$$

That is $z \in \mathcal{K}^*$. □

The result yields further:

Proposition 4 *Let $\mathcal{K} \subset \mathcal{H}$ be a closed convex cone.*

Then

$$\mathcal{K} = \mathcal{K}^{**}.$$

PROOF: It is evident that \mathcal{K} is a subset of \mathcal{K}^{**} .

Let $x \in \mathcal{K}^{**}$. Then, because of Theorem 3, there exist $y \in \mathcal{K}$, $z \in \mathcal{K}^*$ with $x = y + z$ and $\langle y, z \rangle_{\mathcal{H}} = 0$. From that we have

$$0 \geq \langle x, z \rangle_{\mathcal{H}} = \langle y, z \rangle_{\mathcal{H}} + \langle z, z \rangle_{\mathcal{H}} = \|z\|^2.$$

With this we obtain $\|z\|^2 = 0$ meaning $z = 0$. This settles $x = y$ and therefore $\mathcal{K}^{**} \subset \mathcal{K}$. □

Finishing preliminary work, we can now begin to deal with the actual topic: The *orthogonal decomposition with respect to dual cones* proven by J.J.Moreau [2].

Proposition 5 (Moreau)

Let $x, y, z \in \mathcal{H}$. Moreover, let $\mathcal{K}, \mathcal{C} \subseteq \mathcal{H}$ be two closed convex and mutually dual cones. Then the following statements are equivalent:

- (i) $z = x + y$, with $x \in \mathcal{K}$, $y \in \mathcal{C}$ and $\langle x, y \rangle_{\mathcal{H}} = 0$
- (ii) $x = \text{proj}_{\mathcal{K}} z$ $y = \text{proj}_{\mathcal{C}} z$

PROOF: Let $x, y, z \in \mathcal{H}$ and satisfying (i). Since for $y \in \mathcal{C}$ and $\forall p \in \mathcal{K}$ the inequality

$$\langle z - x, p - x \rangle_{\mathcal{H}} = \langle y, p - x \rangle_{\mathcal{H}} = \langle y, p \rangle_{\mathcal{H}} \leq 0 \quad \text{holds,}$$

the characterization (2) of Theorem 1 is valid, that is $x = \text{proj}_{\mathcal{K}} z$. Consider $x \in \mathcal{K}$ and let $q \in \mathcal{C}$. By the same procedure as above, we obtain:

$$\langle z - y, q - y \rangle_{\mathcal{H}} = \langle x, q - y \rangle_{\mathcal{H}} = \langle x, q \rangle_{\mathcal{H}} \leq 0,$$

which means $y = \text{proj}_{\mathcal{C}} z$. Together this yields (ii).

The second step is according to proof of Theorem 3. □

Remark: Let \mathfrak{M} be a closed subspace of \mathcal{H} . Since \mathfrak{M} is closed with respect to the multiplication by scalars, the proposition above is a generalization of the *orthogonal decomposition with respect to \mathfrak{M} and \mathfrak{M}^{\perp}* .

2 Some properties

In this section we want to show some important properties of the decomposition in general and also for Sobolev spaces as some special Hilbert spaces.

Theorem 6 (*Lipschitz continuity*)

Let \mathcal{H} be a Hilbert space, $\mathcal{K} \subset \mathcal{H}$ a closed convex cone and \mathcal{K}^* its dual cone. Moreover, let $u, v \in \mathcal{H}$ arbitrary with the corresponding Moreau decomposition $u = u_1 + u_2, v = v_1 + v_2$ where $u_1, v_1 \in \mathcal{K}, u_2, v_2 \in \mathcal{K}^*$ and $\langle u_1, u_2 \rangle = 0, \langle v_1, v_2 \rangle = 0$.

Then we have that

$$\|u_1 - v_1\|^2 + \|u_2 - v_2\|^2 \leq \|u - v\|^2$$

In particular, the Moreau projection onto \mathcal{K} is Lipschitz continuous with constant 1.

PROOF: Because of the definition of the dual cone \mathcal{K}^* ,

$$\langle u_1, v_2 \rangle \leq 0 \quad \text{und} \quad \langle v_1, u_2 \rangle \leq 0 \text{ hold}$$

This inequality and the assumption of the theorem give

$$\begin{aligned} \|u - v\|^2 &= \langle u_1 + u_2 - v_1 - v_2, u_1 + u_2 - v_1 - v_2 \rangle \\ &= \langle (u_1 - v_1) + (u_2 - v_2), (u_1 - v_1) + (u_2 - v_2) \rangle \\ &= \|u_1 - v_1\|^2 + \|u_2 - v_2\|^2 + 2\langle u_1 - v_1, u_2 - v_2 \rangle \\ &= \|u_1 - v_1\|^2 + \|u_2 - v_2\|^2 + 2\langle u_1, u_2 \rangle + 2\langle v_1, v_2 \rangle - 2\langle u_1, v_2 \rangle - 2\langle v_1, u_2 \rangle \\ &= \|u_1 - v_1\|^2 + \|u_2 - v_2\|^2 - 2\langle u_1, v_2 \rangle - 2\langle v_1, u_2 \rangle \\ &\geq \|u_1 - v_1\|^2 + \|u_2 - v_2\|^2. \end{aligned}$$

□

Now, let us look at the following special situation:

Let $\Omega = B \subset \mathbb{R}^n$ be a ball centered at the origin and $\mathcal{H} = H_0^k(\Omega)$ the Hilbert space equipped with the scalar product

$$\langle u, v \rangle_{H_0^k} = \begin{cases} \int_{\Omega} (\Delta^{k/2} u)(\Delta^{k/2} v) dx, & \text{if } k \text{ even} \\ \int_{\Omega} (\nabla \Delta^{(k-1)/2} u)(\nabla \Delta^{(k-1)/2} v) dx, & \text{if } k \text{ odd.} \end{cases}$$

We consider the closed convex cone

$$\mathcal{K} = \{u \in H_0^k(\Omega) | u(x) \geq 0 \text{ a.e.}\}.$$

It is easy to see that \mathcal{K} is closed. Indeed, let $u \in H_0^k(\Omega) \setminus \mathcal{K}$. For this function there exists a subset $M \subset \Omega$ with positive measure, such that $u(x) < 0$ for all $x \in M$. For each $v \in \mathcal{K}$ we obtain

$$\|u - v\|_{H_0^k(\Omega)} \geq \frac{1}{C} \|u - v\|_{L^2(M)} \geq \|u\|_{L^2(M)} > 0.$$

That means the complement of \mathcal{K} is open, showing the closedness of \mathcal{K} .

The dual cone \mathcal{K}^* in H_0^k consists of all weak subsolutions of the polyharmonic equation under Dirichlet boundary conditions, that is

$$\mathcal{K}^* = \{v \in H_0^k(\Omega) | \langle v, u \rangle_{H_0^k} \leq 0 \forall u \in \mathcal{K}\}.$$

In this case, we can show a further property of the dual cone \mathcal{K}^* , related to differential inequalities and comparison principles:

Lemma 7 *Let $v \in \mathcal{K}^*$, then $v(x) \leq 0$ holds for a.e. $x \in \Omega$, that is $\mathcal{K}^* \subset -\mathcal{K}$.*

PROOF: Let $v \in \mathcal{K}^*$ and $f \in C_0^\infty(\Omega) \cap \mathcal{K}$ be an arbitrary function. Moreover, we consider the following problem:

$$\begin{cases} (-\Delta)^k u = f & \text{in } \Omega \\ D^K u = 0 & \text{on } \partial\Omega \quad K = 0, \dots, k-1; \end{cases}$$

Because of the positivity of the Green function for the Dirichlet problem

$$G_{k,n} = c_{k,n} |x - y|^{2k-n} \int_1^{||x|y-x|/|x-y|} (v^2 - 1)^{k-1} v^{1-n} dv, \quad (5)$$

where $c_{k,n} > 0$ is a well known constant, we have that $u \in \mathcal{K}$. By this we obtain

$$0 \geq \langle v, u \rangle_{H_0^k} = \int_{\Omega} v (-\Delta)^k u = \int_{\Omega} v f.$$

Since the inequality holds for all $f \in C_0^\infty(\Omega) \cap \mathcal{K}$ it follows by density that $\int v f \leq 0$ for all $f \in L^2(\Omega)$ such that $f(x) \geq 0$ a.e.. This shows $v(x) \leq 0$ a.e. and therefore $\mathcal{K}^* \subset -\mathcal{K}$. \square

After having shown these two interesting properties, let us come to the main objective of this paper:

3 Explicit examples

Let us look at some explicit examples. In order to find out how the decomposition works in special situations. Let $\mathcal{H} = H_0^1(-1, 1)$ be the given Hilbert space endowed with the scalar product $\langle u, v \rangle_{\mathcal{H}} = \int_{-1}^1 u'(x)v'(x)dx$. We want to decompose with respect to the closed convex cone of nonnegative functions: $\mathcal{K} = \{v \in H_0^1(-1, 1) : v \geq 0 \text{ a.e.}\}$. The following example is taken from [3].

Example 8 *At first, we will calculate the Moreau decomposition $u = u_1 + u_2$ of a negative peak: Let $\varepsilon \in (0, 1)$.*

$$u_\varepsilon(x) = \begin{cases} \frac{|x|}{\varepsilon} - 1, & \text{if } |x| \leq \varepsilon \\ 0, & \text{else} \end{cases}$$

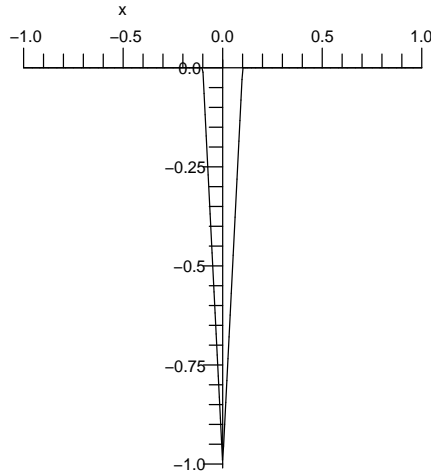


Figure 1: A negative peak: $u_{0.1}(x)$

Furthermore, let $u_2(x) = |x| - 1$ and

$$u_1(x) = u_\varepsilon(x) - u_2(x) = \begin{cases} \frac{|x|}{\varepsilon} - |x|, & \text{if } |x| \leq \varepsilon \\ 1 - |x|, & \text{otherwise.} \end{cases}$$

Consequently, we obtain the following illustration for $u_\varepsilon(x)$:

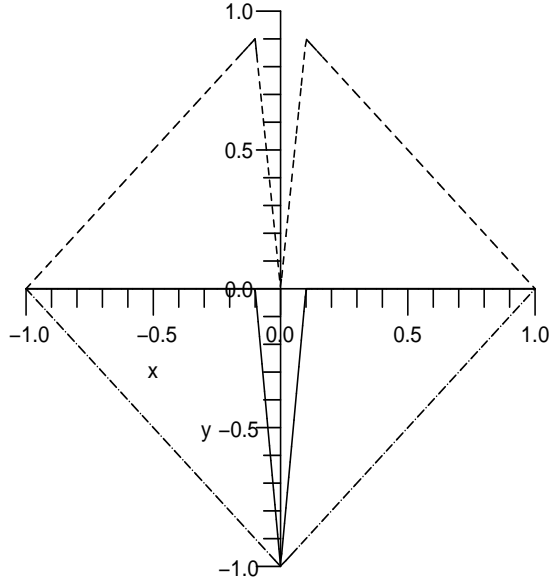


Figure 2: Moreau decomposition of $u_{0.1}(x)$

To prove that $u_1 + u_2$ is the Moreau decomposition of u_ε with respect \mathcal{K} and \mathcal{K}^* we have to show that the following conditions are fulfilled:

$$(\alpha) u_1 \in \mathcal{K}, \quad (\beta) \langle u_2, \varphi \rangle_{\mathcal{H}} \leq 0 \quad \forall \varphi \in \mathcal{K}, \quad (\gamma) \langle u_2, u_1 \rangle_{\mathcal{H}} = 0. \quad (6)$$

because of the definition of $u_1(x)$ (q.v. Fig. 2) we obtain $u_1(x) \geq 0$ and hence $u_1 \in \mathcal{K}$. For all $\varphi \in \mathcal{K}$. We have

$$\langle u_2, \varphi \rangle_{\mathcal{H}} = \int_{-1}^1 u_2'(x) \varphi'(x) dx = - \int_{-1}^0 \varphi'(x) dx + \int_0^1 \varphi'(x) dx = -2\varphi(0) \leq 0,$$

i.e. (β) and hence $u_2 \in \mathcal{K}^*$. By replacing φ by u_1 with $u_1(0) = 0$, we have (γ) . Together, $u_\varepsilon = u_1 + u_2$ is the Moreau decomposition. \square

Proposition 9 The Moreau decomposition in H_0^1 is not continuous with respect to the L^2 -norm.

After this, we want to extend the example and consider two negative peaks instead of one.

Example 10 Let $\varepsilon \in (0, \frac{1}{2})$. Now we want to determine the decomposition for:

$$u_\varepsilon(x) = \begin{cases} -m(\varepsilon)x - n_1(\varepsilon), & \text{if } x \in (\varepsilon - 1, -\frac{1}{2}) \\ m(\varepsilon)x + n_2(\varepsilon), & \text{if } x \in [-\frac{1}{2}, -\varepsilon) \\ -m(\varepsilon)x + n_2(\varepsilon), & \text{if } x \in (\varepsilon, \frac{1}{2}) \\ m(\varepsilon)x - n_1(\varepsilon), & \text{if } x \in [\frac{1}{2}, 1 - \varepsilon) \\ 0, & \text{otherwise} \end{cases} \quad (7)$$

with $m(\varepsilon) = \frac{2}{1-2\varepsilon}$, $n_1(\varepsilon) = 2 \left(\frac{1-\varepsilon}{1-2\varepsilon} \right)$ and $n_2(\varepsilon) = \frac{2\varepsilon}{1-2\varepsilon}$.

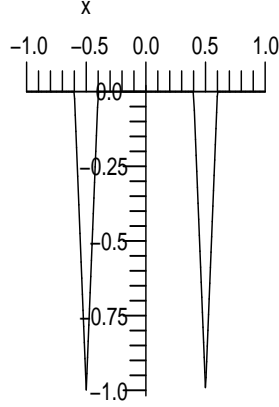


Figure 3: Two negative peaks: $u_{0.4}(x)$

We proceed as in the first Example 8. We claim that

$$u_2(x) = \begin{cases} -2x - 2, & \text{if } x \in (-1, -\frac{1}{2}) \\ -1, & \text{if } |x| \leq \frac{1}{2} \\ 2x - 2, & \text{if } x \in (\frac{1}{2}, 1) \end{cases}$$

$$u_1 = u_\varepsilon - u_2 = \begin{cases} 2x + 2, & \text{if } x \in (-1, \varepsilon - 1) \\ -m_1(\varepsilon)x - n_2(\varepsilon), & \text{if } x \in [\varepsilon - 1, -\frac{1}{2}) \\ m(\varepsilon)x + n_3(\varepsilon), & \text{if } x \in [-\frac{1}{2}, -\varepsilon) \\ 1, & \text{if } x \in [-\varepsilon, \varepsilon] \\ -m(\varepsilon)x + n_3(\varepsilon), & \text{if } x \in (\varepsilon, \frac{1}{2}) \\ m_1(\varepsilon)x - n_2(\varepsilon), & \text{if } x \in [\frac{1}{2}, 1 - \varepsilon) \\ -2x + 2, & \text{if } x \in [1 - \varepsilon, 1) \end{cases}, \quad (8)$$

with the coefficient functions $m_1(\varepsilon) = 2\varepsilon \cdot m(\varepsilon)$, $n_3(\varepsilon) = \frac{m(\varepsilon)}{2}$ and $m(\varepsilon)$, $n_2(\varepsilon)$ defined as in (7).

We have to prove that the conditions (α) - (γ) hold. By looking at equation (8) (or fig.4), we see $u_1 \in \mathcal{K}$. Let $\varphi \in \mathcal{K}$ be arbitrary. We obtain

$$\begin{aligned} \langle u_2, \varphi \rangle_{\mathcal{H}} &= \int_{-1}^1 u_2'(x) \varphi'(x) dx \\ &= -2 \int_{-1}^{-\frac{1}{2}} \varphi'(x) dx + \int_{-\frac{1}{2}}^{\frac{1}{2}} 0 \cdot \varphi'(x) dx + 2 \int_{\frac{1}{2}}^1 \varphi'(x) dx \\ &= -2 \left(\varphi \left(-\frac{1}{2} \right) + \varphi \left(-\frac{1}{2} \right) \right) \leq 0. \end{aligned}$$

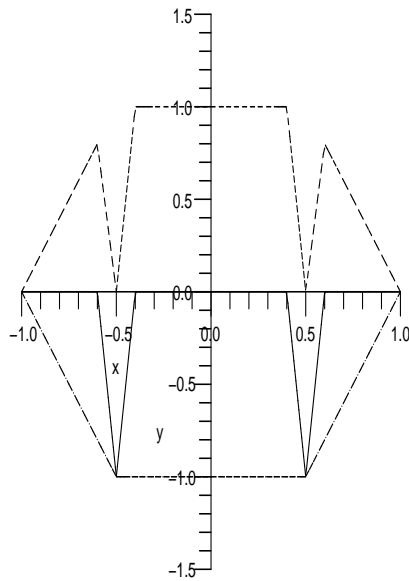


Figure 4: Moreau decomposition of $u_{0.4}(x)$

Hence $u_2 \in \mathcal{K}^*$. Now we replace φ by u_1 and because of $u_1(\pm\frac{1}{2}) = 0$ (γ) is valid. That means $u_1 + u_2$ is the Moreau decomposition of u_ε . \square

Before we continue to give further examples, let us take a closer look at the previous ones. So far, the most important thing was to find a function u_2 . How can such a function be determined?

We try to give a physical justification for the choice of u_2 . Let us assume the following situation: Let be given a rubber band spanned between $(-1,1)$. This leads to the question how the position of the rubber band changes if we had given a negative peak (see fig 2) stretching it downwards. We have seen in Example 8 that the above defined function u_2 is a solution to the problem.

Let us look at the following example with regard to this aspect.

Example 11 We calculate the decomposition for

$$u_k(x) = \begin{cases} k^2 x^2 - 1, & \text{if } |x| \leq \frac{1}{k} \\ 0, & \text{else} \end{cases} \quad \forall k \in \mathbb{N}.$$

Looking at Figures 5-6, under the above-mentioned aspect, we come to the conjecture that

$$u_2(x) = \begin{cases} -a(k)(x+1), & \text{if } x \in (-1, -x_0) \\ k^2 x^2 - 1, & \text{if } |x| \leq x_0 \\ a(k)(x-1), & \text{if } x \in (x_0, 1) \end{cases},$$

with $a(k) = 2k(k - \sqrt{k^2 - 1})$, $x_0 = 1 - \sqrt{1 - \frac{1}{k^2}}$ and

$$u_1(x) = u_k - u_2 = \begin{cases} a(k)(x + 1), & \text{if } x \in (-1, -\frac{1}{k}] \\ k^2x^2 + a(k)(x + 1) - 1, & \text{if } x \in (\frac{1}{k}, -x_0) \\ 0, & \text{if } |x| \leq x_0 \\ k^2x^2 - a(k)(x - 1) - 1, & \text{if } x \in (x_0, \frac{1}{k}) \\ -a(k)(x - 1), & \text{if } x \in [\frac{1}{k}, 1) \end{cases},$$

$a(k)$ and x_0 as above, are the functions we are looking for.

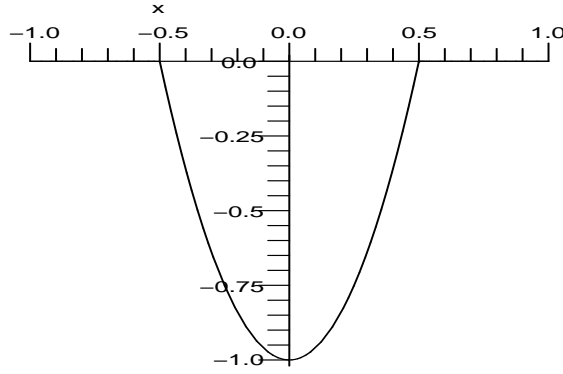


Figure 5: $u_{k=2}(x)$

For an arbitrary $\varphi \in \mathcal{K}$, we obtain by integration by parts:

$$\begin{aligned} \langle u_2, \varphi \rangle_{\mathcal{H}} &= \int_{-1}^1 u_2'(x) \varphi'(x) dx = - \int_{-1}^1 u_2''(x) \varphi(x) dx \\ &= - \left[\int_{-1}^{-x_0} \underbrace{u_2''(x)}_{\equiv 0} \varphi(x) dx + \int_{-x_0}^{x_0} u_2''(x) \varphi(x) dx + \int_{x_0}^1 \underbrace{u_2''(x)}_{\equiv 0} \varphi(x) dx \right] \\ &= - \int_{-x_0}^{x_0} u_2''(x) \varphi(x) dx = -2k^2 \int_{-x_0}^{x_0} \varphi(x) dx \leq 0, \end{aligned}$$

showing that $u_2 \in \mathcal{K}^*$. Obviously $u_1 \in \mathcal{K}$ and since $u_1 \equiv 0$ on $[-x_0, x_0]$ we obtain after substitution of φ , $(\gamma) \langle u_2, u_1 \rangle_{\mathcal{H}} = 0$. This yields: $u_1 + u_2$ is the Moreau decomposition of u_k . \square

Up to now we only considered negative functions and it is an interesting question to study in more detail the influence of positive parts on the function u_2 . The following examples show that the positive parts have no impact on the choice of u_2 , i.e. they only influence the u_1 -part of the decomposition.

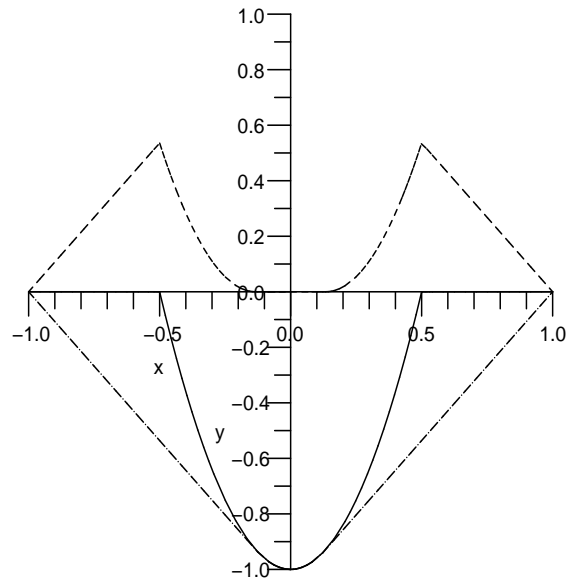


Figure 6: Moreau decomposition of $u_{k=2}(x)$

Example 12 Consider

$$u_\varepsilon(x) = \begin{cases} -m(\varepsilon)x - n_1(\varepsilon), & \text{if } x \in (\varepsilon - 1, -\frac{1}{2}) \\ m(\varepsilon)x + n_2(\varepsilon), & \text{if } x \in [-\frac{1}{2}, -\varepsilon] \\ m(\varepsilon)x - n_2(\varepsilon), & \text{if } x \in (\varepsilon, \frac{1}{2}] \\ -m(\varepsilon)x + n_1(\varepsilon), & \text{if } x \in (\frac{1}{2}, 1 - \varepsilon) \\ 0, & \text{otherwise} \end{cases} \quad \forall \varepsilon \in \left(0, \frac{1}{2}\right), \quad (9)$$

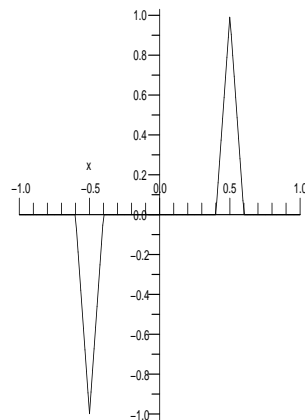


Figure 7: A positive and a negative peak: $u_\varepsilon(x)$

Let the coefficient functions be defined as in Example 10. Furthermore, let

$$u_2(x) = \begin{cases} -2x - 2, & \text{if } x \in (-1, -\frac{1}{2}) \\ \frac{2}{3}(x - 1), & \text{if } x \in (-\frac{1}{2}, 1) \end{cases} \text{ and}$$

$$u_1 = u_\varepsilon - u_2 = \begin{cases} 2x + 2, & \text{if } x \in (-1, \varepsilon - 1) \\ -m_1(\varepsilon)x - n_2(\varepsilon), & \text{if } x \in [\varepsilon - 1, -\frac{1}{2}) \\ m(\varepsilon)x + n_3(\varepsilon), & \text{if } x \in [-\frac{1}{2}, -\varepsilon) \\ -\frac{2}{3}(x - 1), & \text{if } x \in [-\varepsilon, \varepsilon] \\ (m(\varepsilon) - \frac{2}{3})x - n_2(\varepsilon) + \frac{2}{3}, & \text{if } x \in (\varepsilon, \frac{1}{2}) \\ -(m(\varepsilon) + \frac{2}{3})x + n_1(\varepsilon) + \frac{2}{3}, & \text{if } x \in [\frac{1}{2}, 1 - \varepsilon) \\ -\frac{2}{3}(x - 1), & \text{if } x \in [1 - \varepsilon, 1) \end{cases}$$

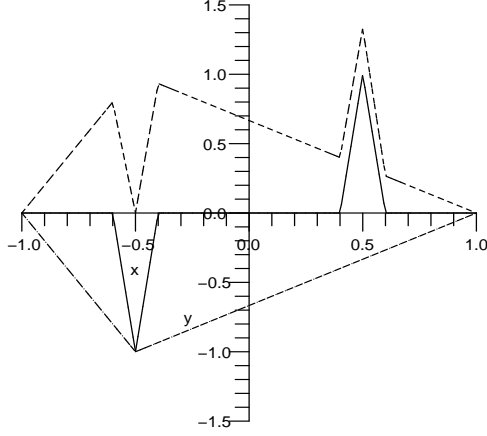


Figure 8: Decomposition of $u_\varepsilon(x)$

It is clear that $u_1 \in \mathcal{K}$. As for (β) : Let $\varphi \in \mathcal{K}$ be, then we obtain by integration by parts

$$\begin{aligned} \langle u_2, \varphi \rangle_{\mathcal{H}} &= \int_{-1}^1 u_2'(x) \varphi'(x) dx \\ &= -2 \int_{-1}^{-\frac{1}{2}} \varphi'(x) dx + \frac{2}{3} \int_{-\frac{1}{2}}^1 \varphi'(x) dx \\ &= -\frac{8}{3} \varphi\left(-\frac{1}{2}\right) \leq 0, \end{aligned}$$

which means that $u_2(x) \in \mathcal{K}^*$. With $u_1(-\frac{1}{2}) = 0$ we have $\langle u_2, u_1 \rangle_{\mathcal{H}} = 0$. This shows that $u_\varepsilon = u_1 + u_2$ is the decomposition of $u_\varepsilon(x)$. \square

After illustrating several examples occurring in the Hilbert space H_0^1 , we now give two examples for the case $u \in H_0^2$.

Example 13 Let $\mathcal{H} = H_0^2(-1, 1)$ be the Hilbert space equipped with the the scalar product

$$\langle u, v \rangle_{\mathcal{H}} = \int_{-1}^1 u''(x)v''(x)dx$$

We will decompose with respect to the cone:

$$\mathcal{K} = \{v \in H_0^2(-1, 1) : v \geq 0 \text{ f..}\}.$$

Consider

$$u(x) = \begin{cases} 256x^6 - 144x^4 + 24x^2 - 1, & \text{if } |x| \leq \frac{1}{2} \\ 0, & \text{otherwise} \end{cases}$$

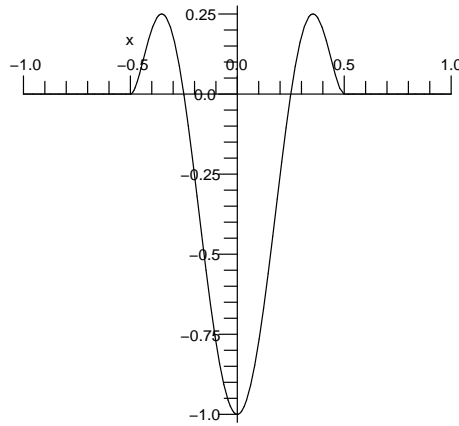


Figure 9: $u(x)$

Moreover, let

$$u_2(x) = \begin{cases} 2x^3 + 3x^2 - 1, & \text{if } x \in (-1, 0] \\ -2x^3 + 3x^2 - 1, & \text{if } x \in (0, 1) \end{cases}$$

and

$$u_1(x) = u(x) - u_2(x) = \begin{cases} -2x^3 - 3x^2 + 1, & \text{if } x \in (-1, -\frac{1}{2}) \\ 256x^6 - 144x^4 - 2x^3 + 21x^2, & \text{if } x \in [-\frac{1}{2}, 0] \\ 256x^6 - 144x^4 + 2x^3 + 21x^2, & \text{if } x \in (0, \frac{1}{2}] \\ 2x^3 - 3x^2 + 1, & \text{if } x \in (\frac{1}{2}, 1) \end{cases}.$$

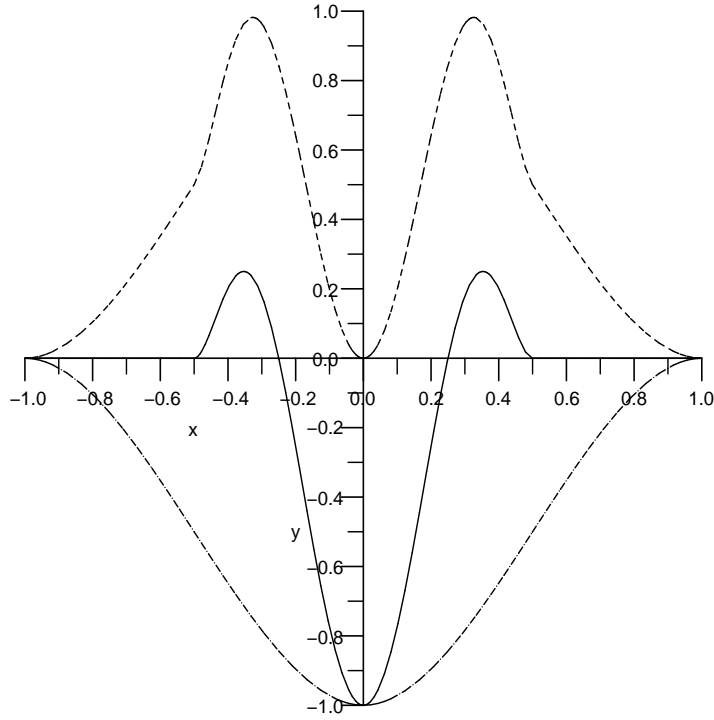


Figure 10: Moreau decomposition of $u(x)$

It is easy to see, that $u_1 \geq 0$ i.e. $u_1 \in \mathcal{K}$. To show that $u_2 \in \mathcal{K}^*$, we consider an arbitrary $\varphi \in \mathcal{K}$. We obtain, after integration by parts:

$$\begin{aligned}
\langle u_2, \varphi \rangle_{H_0^2} &= \int_{-1}^0 u_2''(x) \varphi''(x) dx + \int_0^1 u_2''(x) \varphi''(x) dx \\
&= u_2''(0) \varphi'(0) - u_2''(-1) \varphi'(-1) - \int_{-1}^0 u_2'''(x) \varphi'(x) dx \\
&\quad + u_2''(1) \varphi'(1) - u_2''(0) \varphi'(0) - \int_0^1 u_2'''(x) \varphi'(x) dx \\
&= -12 \int_{-1}^0 \varphi'(x) dx + 12 \int_0^1 \varphi'(x) dx = -24 \varphi(0) \\
&\leq 0,
\end{aligned}$$

i.e. $u_2 \in \mathcal{K}^*$. Now we replace φ by u_1 and we conclude $\langle u_2, u_1 \rangle_{\mathcal{H}} = 0$. This shows that $u = u_1 + u_2$ is the Moreau decomposition. \square

The physical interpretation agrees with the one given above, we have just replaced the rubberband by an elastic bar clamped over the interval $(-1,1)$. Having this in mind we consider the following:

Example 14 We want to decompose in the Hilbert space $\mathcal{H} = H_0^2(-2,2)$, endowed

with the scalar product

$$\langle u, v \rangle_{\mathcal{H}} = \int_{-2}^2 u''(x)v''(x)dx.$$

As above we consider the cone of the nonnegative functions :

$$\mathcal{K} = \{v \in H_0^2(-2, 2) : v \geq 0 \text{ f..}\}.$$

Let

$$u(x) = \begin{cases} -\frac{16}{25}x^6 + \frac{72}{25}x^4 - \frac{81}{25}x^2, & \text{if } |x| \leq \frac{3}{2} \\ 0, & \text{sonst} \end{cases}$$

be the given function.

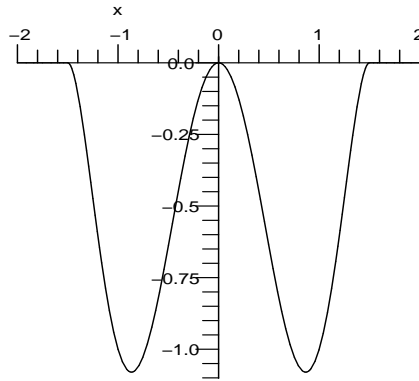


Figure 11: The function $u(x) \in H_0^2(-2, 2)$

Furthermore, Let

$$u_2(x) = \begin{cases} \frac{4}{5}x^3 + 3x^2 + \frac{12}{5}x - \frac{4}{5}, & \text{if } x \in (-2, -1) \\ \frac{3}{5}x^2 - \frac{8}{5}, & \text{if } |x| \leq 1 \\ -\frac{4}{5}x^3 + 3x^2 - \frac{12}{5}x - \frac{4}{5}, & \text{if } x \in (1, 2) \end{cases}$$

and therefore

$$u_1(x) = u(x) - u_2(x) = \begin{cases} -\frac{4}{5}x^3 - 3x^2 - \frac{12}{5}x + \frac{4}{5}, & \text{if } x \in (-2, -\frac{3}{2}) \\ -\frac{16}{25}x^6 + \frac{72}{25}x^4 - \frac{4}{5}x^3 - \frac{156}{25}x^2 - \frac{12}{5}x + \frac{4}{5}, & \text{if } x \in [-\frac{3}{2}, -1) \\ -\frac{16}{25}x^6 + \frac{72}{25}x^4 - \frac{96}{25}x^2 + \frac{8}{5}, & \text{if } |x| \leq 1 \\ -\frac{16}{25}x^6 + \frac{72}{25}x^4 + \frac{4}{5}x^3 - \frac{156}{25}x^2 + \frac{12}{5}x + \frac{4}{5}, & \text{if } x \in (1, \frac{3}{2}] \\ \frac{4}{5}x^3 - 3x^2 + \frac{12}{5}x + \frac{4}{5}, & \text{if } x \in (\frac{3}{2}, 2) \end{cases}.$$

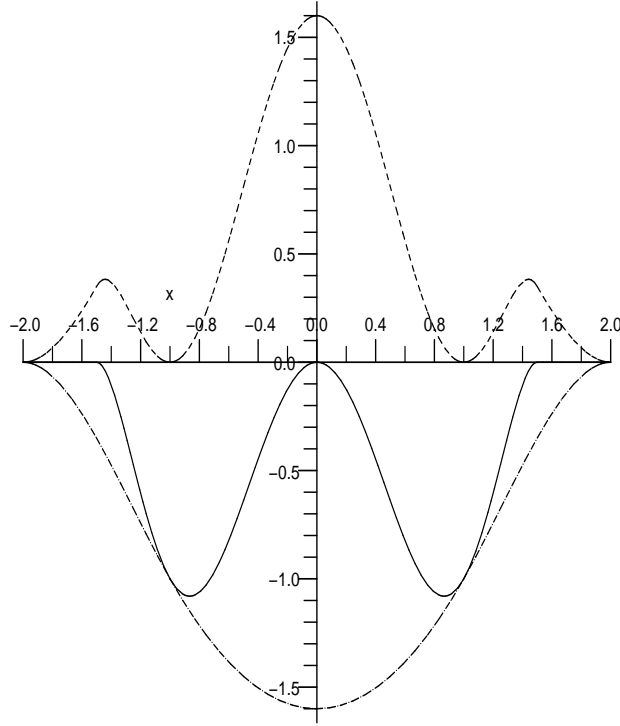


Figure 12: Moreau decomposition of $u(x)$

By looking at the figure or the definition of u_1 , we have that $u_1 \in \mathcal{K}$. Again it remains to prove $u_2 \in \mathcal{K}^*$. To this end, let $\varphi \in \mathcal{K}$ be arbitrary. Then we obtain

$$\begin{aligned}
\langle u_2, \varphi \rangle_{H_0^2} &= \int_{-2}^{-1} u_2''(x) \varphi''(x) dx + \int_{-1}^1 u_2''(x) \varphi''(x) dx + \int_1^2 u_2''(x) \varphi''(x) dx \\
&= u_2''(-1) \varphi'(-1) - u_2''(-2) \varphi'(-2) - \int_{-2}^{-1} u_2'''(x) \varphi'(x) dx \\
&\quad + u_2''(1) \varphi'(1) - u_2''(-1) \varphi'(-1) - \int_{-1}^1 u_2'''(x) \varphi'(x) dx \\
&\quad + u_2''(2) \varphi'(2) - u_2''(1) \varphi'(1) - \int_1^2 u_2'''(x) \varphi'(x) dx \\
&= -\frac{24}{5} \int_{-2}^{-1} \varphi'(x) dx - \int_{-1}^1 0 \cdot \varphi'(x) dx + \frac{24}{5} \int_1^2 \varphi'(x) dx \\
&= -\frac{24}{5} (\varphi(-1) + \varphi(1)) \leq 0,
\end{aligned}$$

proving $u_2 \in \mathcal{K}^*$. Because of $u_1 \in \mathcal{K}$ with $u_1(\pm 1) = 0$ we have $\langle u_2, u_1 \rangle_{\mathcal{H}} = 0$. All this together means that $u_1 + u_2$ is the Moreau decomposition of u in the Hilbert space $H_0^2(-2, 2)$. \square

4 Connection with obstacle problems

Our aim now is to characterize the Moreau decomposition $u = u_1 + u_2$ for a given function $u \in H_0^1$ by means of a minimization problem. In this context, a variational inequality for this problem shall be presented. Finally, we want to define the obstacle problem, induced by the minimization problem, for general situations, and we will give some fundamental results related to this topic. See also [5].

Let again be $\mathcal{H} = H_0^1((\alpha, \beta))$, $\alpha, \beta \in \mathbb{R}$ the Hilbert space equipped with the scalar product

$$\langle u, v \rangle_{\mathcal{H}} = \int_{\alpha}^{\beta} u'(x)v'(x)dx$$

and let

$$\mathcal{K} = \{v \in H_0^1((\alpha, \beta)) : v \geq 0 \text{ a.e.}\}$$

be the convex cone with respect to which we want to perform the decomposition.

As we have seen in Section 3, the computation of u_2 and u_1 resp., is equivalent to the minimization problem:

Let $u \in \mathcal{H}$ be given. Find $u_1 \in \mathcal{K}$ such that $\|u - u_1\|_{\mathcal{H}} = \|u_2\|_{\mathcal{H}}$ is minimal under the constraint that $u_2 \leq u$ a.e. (with $u_2 \in \mathcal{K}^* \subset -\mathcal{K}$). Thus, we look for

$$u_2 \in \mathcal{K}_u := \{v \in \mathcal{K}^* : v \leq u \text{ a.e.}\} : \int_{\alpha}^{\beta} |\nabla u_2(x)|^2 dx = \min! \quad (10)$$

This type of minimization problem is called an *obstacle problem*, because the constraint $u_2 \leq u$ a.e. constitutes an obstacle.

Lemma 15 *The absolute minima $u_2 \in \mathcal{K}_u$ in (10) are solutions to the following variational inequality:*

$$\langle u_2, v - u_2 \rangle_{\mathcal{H}} \geq 0 \quad \forall v \in \mathcal{K}_u. \quad (11)$$

PROOF: Let u_2 be an absolute minimum of $v \in \mathcal{K}_u$. Since \mathcal{K}_u is a convex subset of \mathcal{H} , the convex combination $(1-t)u_2 + tv \in \mathcal{K}_u$ ($0 < t \leq 1$). Then we have

$$\begin{aligned} \int_{\alpha}^{\beta} |\nabla u_2(x)|^2 dx &\leq \int_{\alpha}^{\beta} |\nabla((1-t)u_2(x) + tv(x))|^2 dx \\ &= \int_{\alpha}^{\beta} |\nabla u_2(x) + t\nabla(v - u_2)(x)|^2 dx \\ &= \int_{\alpha}^{\beta} |\nabla u_2(x)|^2 + 2t \int_{\alpha}^{\beta} \nabla u_2(x)\nabla(v - u_2)(x)dx + t^2 \int_{\alpha}^{\beta} |\nabla(v - u_2)(x)|^2 dx \end{aligned}$$

After dividing by t and for $t \searrow 0$ we obtain

$$0 \leq \int_{\alpha}^{\beta} \nabla u_2 \nabla(v - u_2) dx = \langle u_2, v - u_2 \rangle_{\mathcal{H}}, \quad \forall v \in \mathcal{K}_u,$$

i.e. the absolute minimum u_2 satisfies the variational inequality. □

Next, we want to formulate a general obstacle problem.

Let $\Omega \subset \mathbb{R}^n$ be bounded connected and having a smooth boundary. Moreover, let $a_{ij} \in L^\infty(\Omega)$ satisfy

$$(1/\Lambda)\xi^2 \leq a_{ij}(x)\xi_i\xi_j \leq \Lambda\xi^2 \quad \text{a.e.} \quad \xi \in \mathbb{R}^n,$$

where Λ is an ellipticity constant. We define the mapping

$$L : H_0^1(\Omega) \rightarrow H^{-1}(\Omega) \quad \text{with} \quad \langle Lu, v \rangle = a(u, v) \quad u, v \in H_0^1(\Omega,)$$

where for the $a(u, v)$ we have

$$a(u, v) = \int_{\Omega} a_{ij}(x)u_{x_j}(x)v_{x_i}(x)dx \quad u, v \in H^1(\Omega).$$

We now consider an "obstacle" $\psi \in H^1(\Omega)$ which in addition satisfies $\psi \leq 0$ on $\partial\Omega$. Furthermore, let $\mathcal{K}_\psi = \{v \in H_o^1(\Omega) : v(x) \geq \psi(x) \text{ a.e.}\}$.

With all these assumptions, we can define the obstacle problem as follows:

Problem 1: Let $f \in H^{-1}$ be given. Find $u \in \mathcal{K}_\psi$ so, that

$$a(v, v - u) \geq \langle f, v - u \rangle \quad \text{holds for all } v \in \mathcal{K}_\psi. \quad (12)$$

This gives rise to the question about existence and uniqueness of a solution to (12). This is answered by

Theorem 16 *There exists a unique solution $u \in \mathcal{K}_\psi$ to Problem 1.*

The proof can be found in [5].

With this result at hand, we can investigate the case that we had found a solution $u \in \mathcal{K}_\psi$ for a given obstacle ψ in (12). In this case, we can divide the set Ω into two subsets

- (i) $G := \{x \in \Omega : u(x) > \psi(x)\}$ (open)
- (ii) $I = I[u] = \Omega \setminus G = \{x \in \Omega : u(x) = \psi(x)\}$, (closed).

Definition 17 *The set I is called the coincidence set of the solution u .*

We conclude this seminar paper with the following theorem:

Theorem 18 *Let $u \in \mathcal{K}_\psi$ be a solution of Problem 1. Then there exists a nonnegative Radon measure μ such that*

$$Lu = f + \mu \quad \text{in } \Omega$$

with

$$\text{supp } \mu \subset I = \{x \in \Omega : u(x) = \psi(x)\}$$

In particular,

$$Lu = f \quad \text{in } \Omega \setminus I.$$

References

- [1] K.Deckelnick, *Lineare Funktionalanalysis*,LECTURE NOTES (2003/04)
- [2] J.J. Moreau, *Décomposition orthogonale d'un espace hilbertien selon deux cônes mutuellement polaires*, C. R. ACAD. SCI. PARIS, **255**, 238-240 (1962)
- [3] H.-Ch.Grunau, *Lipschitz continuity of the decomposition with respect to dual cones*, SCRIPT (2004)
- [4] F. Gazzola, H.-Ch.Grunau, *Critical dimensions and higher order Sobolev inequalities with remainder terms*, NODEA NONLINEAR DIFFERENTIAL EQUATIONS APPL. **8**, 35-44 (2001)
- [5] D. Kinderlehrer, G. Stampacchia, *An Introduction to Variational Inequalities and Their Applications*, ACADEMIC PRESS (1980)